ON CERTAIN APPROXIMATE FORMULÆ FOR CALCULATING THE TRAJECTORIES OF SHOT.

IN the postscript to a paper by Mr. W. D. Niven, "On the Calculation of the Trajectories of Shot," which is published in the Proceedings of the Royal Society, vol. xxvi. pp. 268–287, I have given, without demonstration, some convenient and not inelegant formulæ applicable to a limited arc of a trajectory when the resistance is supposed to vary as the nth power of the velocity.

In these formulæ, the angle between the chord of the arc and the tangent at any point is supposed to be always small. The index n is not restricted to integral values, but may take any value whatever.

As the proof of these formulæ is not altogether obvious, and a similar method of treatment may be found useful in other problems, I think it may not be unacceptable to your readers if I show here how the formulæ may be demonstrated.

Analysis.

Investigation of formulæ applicable to a small arc of a trajectory, when the resistance varies as the nth power of the velocity.

Let x and y denote the horizontal and vertical co-ordinates at time t, u the horizontal velocity, and ϕ the angle which the direction of motion makes with the horizon at the same time.

Hence the velocity at time t is $u \sec \phi$, and we may denote the resistance by $ku^n(\sec \phi)^n$, where k is constant throughout the small arc in question.

Also let p and q denote the values of u at the beginning and end of the arc, α and β the corresponding values of ϕ, g the force of gravity, T the time taken to describe the arc, X and Y the corresponding total horizontal and vertical motion.

Making ϕ the independent variable, the fundamental formulæ are—

$$(1)\quad \frac{du}{d\phi} = \frac{ku^{n+1}}{g}(\sec \phi)^{n+1};$$

$$(2)\quad \frac{dx}{d\phi} = -\frac{u^2}{g}(\sec \phi)^2;$$

$$(3)\quad \frac{dy}{d\phi} = -\frac{u^2}{g}(\sec \phi)^2 \tan \phi;$$

$$(4)\quad \frac{dt}{d\phi} = -\frac{u}{g}(\sec \phi)^2.$$

From the first of these equations—

$$\frac{1}{u^{n+1}}\frac{du}{d\phi} = \frac{k}{g}(\sec \phi)^{n+1};$$

and therefore, by integration between the limits $\phi = \alpha$ and $\phi = \beta$,

$$\frac{1}{q^n} - \frac{1}{p^n} = \frac{kn}{g}\int_\beta^\alpha (\sec \phi)^{n+1} d\phi.$$

Also, we have—

$$\mathrm{X} = \frac{1}{g}\int_\beta^\alpha u^2(\sec \phi)^2 d\phi;$$

$$\mathrm{Y} = \frac{1}{g}\int_\beta^\alpha u^2(\sec \phi)^2 \tan \phi\, d\phi;$$

and

$$\mathrm{T} = \frac{1}{g}\int_\beta^\alpha u(\sec \phi)^2 d\phi;$$

and we wish to compare the two former of these definite integrals with the following known one, viz. :—

$$\frac{1}{q^{n-2}} - \frac{1}{p^{n-2}} = (n-2)\int_\beta^\alpha \frac{1}{u^{n-1}}\frac{du}{d\phi}d\phi = \frac{k(n-2)}{g}\int_\beta^\alpha u^2(\sec \phi)^{n+1} d\phi;$$

and the last with—

$$\frac{1}{q^{n-1}} - \frac{1}{p^{n-1}} = (n-1)\int_\beta^\alpha \frac{1}{u^n}\frac{du}{d\phi}d\phi = \frac{k(n-1)}{g}\int_\beta^\alpha u(\sec \phi)^{n+1} d\phi.$$

This may be done by means of the following lemma, which follows immediately from Taylor's theorem :—

Lemma.

If F(ϕ) be any function either of ϕ only, or of ϕ and u, where u is a function of ϕ given by the above differential equation (1), and if α and β be the limiting values of ϕ in the integral and $\gamma = \frac{1}{2}(\alpha + \beta)$, then, putting for a moment $\phi = \gamma + \omega$,

$$\int_\beta^\alpha \mathrm{F}(\phi)d\phi = \int_{-\frac{1}{2}(\alpha-\beta)}^{\frac{1}{2}(\alpha-\beta)} \mathrm{F}(\gamma+\omega)d\omega = \int_{-\frac{1}{2}(\alpha-\beta)}^{\frac{1}{2}(\alpha-\beta)} \left\{ \mathrm{F}(\gamma) + \mathrm{F}'(\gamma)\omega + \mathrm{F}''(\gamma)\frac{\omega^2}{2} + \mathrm{F}'''(\gamma)\frac{\omega^3}{6} + \mathrm{F}''''(\gamma)\frac{\omega^4}{24} + \&\text{c.} \right\} d\omega$$

$$= (\alpha-\beta)\left\{ \mathrm{F}(\gamma) + \frac{1}{24}(\alpha-\beta)^2\mathrm{F}''(\gamma) + \frac{1}{1920}(\alpha-\beta)^4\mathrm{F}''''(\gamma) + \&\text{c.} \right\}$$

where $\mathrm{F}'(\phi) = \frac{d\mathrm{F}(\phi)}{d\phi}$, $\mathrm{F}''(\phi) = \frac{d^2\mathrm{F}(\phi)}{d\phi^2}$, &c., and F($\gamma$), F′($\gamma$), F″($\gamma$), &c., are what F($\phi$), F′($\phi$), F″($\phi$), &c., become when γ is substituted for ϕ, and the corresponding value of u (u_0 suppose) is put for u.

In what follows, the last of the terms above written, which is of the 5th order in ($\alpha - \beta$), is neglected, together with all terms of the same order of small quantities.

All the definite integrals with which we are here concerned are included in the two forms

$$\int_\beta^\alpha u^l(\sec \phi)^m d\phi, \text{ and } \int_\beta^\alpha u^l(\sec \phi)^m \tan \phi\, d\phi.$$

In the first place, we will apply the above formula to the case in which F(ϕ) is a function of ϕ only, viz. when $\mathrm{F}(\phi) = (\sec \phi)^{n+1}$.

Hence

$$\mathrm{F}'(\phi) = (n+1)(\sec \phi)^{n+1} \tan \phi;$$

$$\mathrm{F}''(\phi) = (n+1)[(n+1)(\sec \phi)^{n+1}(\tan \phi)^2 + (\sec \phi)^{n+3}]$$

$$= (n+1)[\overline{n+2}(\sec \phi)^{n+3} - \overline{n+1}(\sec \phi)^{n+1}];$$

and therefore,

$$\int_\beta^\alpha (\sec \phi)^{n+1} d\phi = (\alpha-\beta)(\sec \gamma)^{n+1}\left\{1 + \frac{n+1}{24}(\alpha-\beta)^2[\overline{n+2}(\sec \gamma)^2 - \overline{n+1}]\right\},$$ to the 4th order inclusive.

Hence

$$\frac{1}{q^n} - \frac{1}{p^n} = \frac{kn}{g}(\alpha-\beta)(\sec \gamma)^{n+1}\left\{1 + \frac{n+1}{24}(\alpha-\beta)^2[\overline{n+2}(\sec \gamma)^2 - \overline{n+1}]\right\},$$

which gives q when p is known.

In the next place, let $\mathrm{F}(\phi) = u^l(\sec \phi)^m$.

Hence

$$\mathrm{F}'(\phi) = \frac{d\mathrm{F}\phi}{d\phi} = lu^{l-1}\frac{du}{d\phi}(\sec \phi)^m + mu^l(\sec \phi)^m \tan \phi$$

$$= \mathrm{F}(\phi)\left[\frac{l}{u}\frac{du}{d\phi} \div m \tan \phi\right],$$

or

$$\mathrm{F}'(\phi) = \mathrm{F}(\phi)\left[\frac{kl}{g}u^n(\sec \phi)^{n+1} + m \tan \phi\right];$$

and $F''(\phi) = F'(\phi)\left[\frac{kl}{g}u^n(\sec\phi)^{n+1} + m\tan\phi\right] + F(\phi)\left[\frac{kln}{g}u^{n-1}\frac{du}{d\phi}(\sec\phi)^{n+1} + \frac{kl}{g}(n+1)u^n(\sec\phi)^{n+1}\tan\phi + m(\sec\phi)^2\right]$,

or

$$F''(\phi) = F(\phi)\left[\frac{k^2l^2}{g^2}u^{2n}(\sec\phi)^{2n+2} + 2\frac{klm}{g}u^n(\sec\phi)^{n+1}\tan\phi + m^2(\sec\phi)^2 - m^2\right]$$

$$+ F(\phi)\left[\frac{k^2ln}{g^2}u^{2n}(\sec\phi)^{2n+2} + \frac{kl}{g}(n+1)u^n(\sec\phi)^{n+1}\tan\phi + m(\sec\phi)^2\right]$$

$$= F(\phi)\left\{\frac{k^2l}{g^2}(l+n)u^{2n}(\sec\phi)^{2n+2} + \frac{kl}{g}(2m+n+1)u^n(\sec\phi)^{n+1}\tan\phi + m(m+1)(\sec\phi)^2 - m^2\right\}.$$

Since

$$\frac{du}{d\phi} = \frac{k}{g}u^{n+1}(\sec\phi)^{n+1},$$

this last expression may be put under the form—

$$F''(\phi) = F(\phi)\left\{l(l+n)\left(\frac{du}{ud\phi}\right)^2 + l(2m+n+1)\left(\frac{du}{ud\phi}\right)\tan\phi + m(m+1)(\sec\phi)^2 - m^2\right\}.$$

Hence, by the above lemma,

$$\int_\beta^\alpha u^l\sec\phi^m d\phi = (\alpha-\beta)F(\gamma)\left\{1 + \tfrac{1}{24}(\alpha-\beta)^2\left[l(l+n)\left(\frac{du}{ud\phi}\right)_0^2 + l(2m+n+1)\left(\frac{du}{ud\phi}\right)_0\tan\gamma + m(m+1)(\sec\gamma)^2 - m^2\right]\right\}$$

$$= (\alpha-\beta)\,u_0^l(\sec\gamma)^m\left\{1 + \tfrac{1}{24}(\alpha-\beta)^2(\text{as before})\right\}$$

where $\left(\frac{du}{ud\phi}\right)_0$ denotes what $\frac{du}{ud\phi}$ becomes when $\omega = 0$, or when γ is substituted for ϕ, and u_0 for u, that is—

$$\left(\frac{du}{ud\phi}\right)_0 = \frac{k}{g}u_0^n(\sec\gamma)^{n+1}$$

The factor u_0^l may be eliminated from this expression, and the expression itself simplified, by means of the formula—

$$\frac{1}{q^{n-l}} - \frac{1}{p^{n-l}} = (n-l)\int_\beta^\alpha \frac{1}{u^{n-l+1}}\frac{du}{d\phi}d\phi = \frac{k(n-l)}{g}\int_\beta^\alpha u^l(\sec\phi)^{n+1}d\phi,$$

for, putting $m = n + 1$ in the above expression, we have—

$$\int_\beta^\alpha u^l(\sec\phi)^{n+1}d\phi = (\alpha-\beta)u_0^l(\sec\gamma)^{n+1}\left\{1 + \tfrac{1}{24}(\alpha-\beta)^2\left[l(l+n)\left(\frac{du}{ud\phi}\right)_0^2 + 3l(n+1)\left(\frac{du}{ud\phi}\right)_0\tan\gamma + \overline{n+1}\,\overline{n+2}(\sec\gamma)^2 - (n+1)^2\right]\right\}.$$

Hence

$$\int_\beta^\alpha u^l(\sec\phi)^m d\phi \div \int_\beta^\alpha u^l(\sec\phi)^{n+1}d\phi, \text{ or } \int_\beta^\alpha u^l(\sec\phi)^m d\phi \div \frac{g}{k(n-l)}\left(\frac{1}{q^{n-l}} - \frac{1}{p^{n-l}}\right)$$

$$= (\sec\gamma)^{m-n-1}\left\{1 + \tfrac{1}{24}(\alpha-\beta)^2\left[2l(m-n-1)\left(\frac{du}{ud\phi}\right)_0\tan\gamma + \overline{m-n-1}\,\overline{m+n+2}(\sec\gamma)^2 - \overline{m-n-1}\,\overline{m+n+1}\right]\right\}.$$

It will be noticed that the term involving $\left(\frac{du}{ud\phi}\right)_0^2$ has disappeared by this division.

Now make $m = 2$, and this formula becomes—

$$\int_\beta^\alpha u^l(\sec\phi)^2 d\phi = \frac{g}{k(n-l)}\left(\frac{1}{q^{n-l}} - \frac{1}{p^{n-l}}\right)(\cos\gamma)^{n-1}\left\{1 - \tfrac{1}{24}(\alpha-\beta)^2\left[2l(n-1)\left(\frac{du}{ud\phi}\right)_0\tan\gamma + \overline{n-1}\,\overline{n+4}(\sec\gamma)^2 - \overline{n-1}\,\overline{n+3}\right]\right\}.$$

Divide throughout by g, and put $l = 2$, then, from before,

$$X = \frac{1}{k(n-2)}\left(\frac{1}{q^{n-2}} - \frac{1}{p^{n-2}}\right)(\cos\gamma)^{n-1}\left\{1 - \frac{n-1}{24}(\alpha-\beta)^2\left[4\left(\frac{du}{ud\phi}\right)_0\tan\gamma + (n+4)(\sec\gamma)^2 - \overline{n+3}\right]\right\}.$$

Similarly, divide throughout by g, and put $l = 1$, then—

$$T = \frac{1}{k(n-1)}\left(\frac{1}{q^{n-1}} - \frac{1}{p^{n-1}}\right)(\cos\gamma)^{n-1}\left\{1 - \frac{n-1}{24}(\alpha-\beta)^2\left[2\left(\frac{du}{ud\phi}\right)_0\tan\gamma + (n+4)(\sec\gamma)^2 - \overline{n+3}\right]\right\}$$

Lastly, let

$$F(\phi) = u^l(\sec\phi)^m\tan\phi = f(\phi)\tan\phi \text{ suppose,}$$

so that

$$f(\phi) = u^l(\sec\phi)^m;$$

then

$$F'(\phi) = f'(\phi)\tan\phi + f(\phi)(\sec\phi)^2,$$

and

$$F''(\phi) = f''(\phi)\tan\phi + 2f'(\phi)(\sec\phi)^2 + 2f(\phi)(\sec\phi)^2\tan\phi.$$

Hence

$$\int_\beta^\alpha F(\phi)d\phi = (\alpha-\beta)\{F(\gamma) + \tfrac{1}{24}(\alpha-\beta)^2F''(\gamma)\} \text{ approximately,}$$

$$= (\alpha-\beta)\left\{f(\gamma)\tan\gamma + \tfrac{1}{24}(\alpha-\beta)^2[f''(\gamma)\tan\gamma + 2f'(\gamma)(\sec\gamma)^2 + 2f(\gamma)(\sec\gamma)^2\tan\gamma]\right\};$$

also

$$\int_\beta^\alpha f(\phi)d\phi = (\alpha-\beta)\{f(\gamma) + \tfrac{1}{24}(\alpha-\beta)^2f''(\gamma)\} \text{ approximately;}$$

and therefore

$$\int_\beta^\alpha F(\phi)d\phi \div \int_\beta^\alpha f(\phi)d\phi = \tan\gamma + \tfrac{1}{12}(\alpha-\beta)^2\left[\frac{f'(\gamma)}{f(\gamma)}(\sec\gamma)^2 + (\sec\gamma)^2\tan\gamma\right];$$

in which the term involving $f''(\gamma)$ has disappeared.

Now, since $f(\phi) = u^l(\sec\phi)^m$, we have, as before

$$f'(\phi) = f(\phi)\left[l\left(\frac{du}{ud\phi}\right) + m\tan\phi\right];$$

and therefore—

$$\frac{f'(\gamma)}{f(\gamma)} = l\left(\frac{du}{ud\phi}\right)_0 + m\tan\gamma.$$

Hence—

$$\int_\beta^\alpha F(\phi)d\phi \div \int_\beta^\alpha f(\phi)d\phi = \tan\gamma + \tfrac{1}{12}(\alpha-\beta)^2(\sec\gamma)^2\left[l\left(\frac{du}{ud\phi}\right)_0 + \overline{m+1}\tan\gamma\right];$$

and in the particular case where $l = 2$, and $m = 2$, we have—

$$\frac{Y}{X} = \tan\gamma + \tfrac{1}{12}(\alpha-\beta)^2(\sec\gamma)^2\left[2\left(\frac{du}{ud\phi}\right)_0 + 3\tan\gamma\right]$$

$$= \tan\left\{\gamma + \tfrac{1}{12}(\alpha-\beta)^2\left[2\left(\frac{du}{ud\phi}\right)_0 + 3\tan\gamma\right]\right\}.$$

Hence the angle which the chord of the arc makes with the axis of x is—

$$\gamma + \tfrac{1}{12}(\alpha-\beta)^2\left[2\left(\frac{du}{ud\phi}\right)_0 + 3\tan\gamma\right] = \bar{\gamma}, \text{ suppose.}$$

Multiplying by the value of X found above, we have—

$$Y = \frac{1}{k(n-2)}\left(\frac{1}{q^{n-2}} - \frac{1}{p^{n-2}}\right)(\cos\gamma)^{n-1}\left\{\tan\gamma - \tfrac{1}{24}(\alpha-\beta)^2\left\{\left(\frac{du}{ud\phi}\right)_0\left[4\overline{n-1}(\tan\gamma)^2 - 4(\sec\gamma)^2\right] + \tan\gamma\left[\overline{n-1}\,\overline{n+4}(\sec\gamma)^2 - 6(\sec\gamma)^2 - \overline{n-1}\;n+3\right]\right\}\right\};$$

or

$$Y = \frac{1}{k(n-2)}\left(\frac{1}{q^{n-2}} - \frac{1}{p^{n-2}}\right)(\cos\gamma)^{n-1}\left\{\tan\gamma - \tfrac{1}{24}(\alpha-\beta)^2\left\{\left(\frac{du}{ud\phi}\right)_0\left[4\overline{n-2}(\sec\gamma)^2 - 4\overline{n-1}\right] + \tan\gamma\left[\overline{n-2}\,\overline{n+5}(\sec\gamma)^2 - \overline{n-1}\;n+3\right]\right\}\right\}.$$

Considering $\frac{1}{q^{n-2}} - \frac{1}{p^{n-2}}$, $\frac{1}{q^{n-1}} - \frac{1}{p^{n-1}}$, and $\alpha - \beta$ to be small quantities of the first order, the above expressions for $\frac{1}{q^n} - \frac{1}{p^n}$, X, Y, and T are true to the fourth order.

The quantity $\left(\frac{du}{ud\phi}\right)_0$ which occurs as a factor in some of the terms of the third order may be put under a very convenient form in the following manner.

We have, by Taylor's theorem,

$$u = (u_0) + \left(\frac{du}{d\phi}\right)_0\omega + \left(\frac{d^2u}{d\phi^2}\right)_0\frac{\omega^2}{2} + \&c.$$

In this make $\omega = \frac{1}{2}(\alpha-\beta)$ and $-\frac{1}{2}(\alpha-\beta)$ successively; therefore

$$p = u_0 + \tfrac{1}{2}(\alpha-\beta)\left(\frac{du}{d\phi}\right)_0 + \tfrac{1}{8}(\alpha-\beta)^2\left(\frac{d^2u}{d\phi^2}\right)_0 + \&c.,$$

and

$$q = u_0 - \tfrac{1}{2}(\alpha-\beta)\left(\frac{du}{d\phi}\right)_0 + \tfrac{1}{8}(\alpha-\beta)^2\left(\frac{d^2u}{d\phi^2}\right)_0 - \&c.$$

Hence we have to the first order of small quantities—

$$\frac{p-q}{\alpha-\beta} = \left(\frac{du}{d\phi}\right)_0,$$

and

$$\tfrac{1}{2}(p+q) = u_0;$$

and therefore

$$\left(\frac{du}{ud\phi}\right)_0 = \frac{2(p-q)}{(p+q)(\alpha-\beta)} \text{ to the first order.}$$

Making this substitution for $\left(\frac{du}{ud\phi}\right)_0$ the expressions for X, Y, and T become—

$$X = \frac{1}{k(n-2)}\left(\frac{1}{q^{n-2}} - \frac{1}{p^{n-2}}\right)(\cos\gamma)^{n-1}\left\{1 - \frac{n-1}{3}\cdot\frac{p-q}{p+q}(\alpha-\beta)\tan\gamma - \frac{n-1}{24}(\alpha-\beta)^2\left[n+4(\sec\gamma)^2 - \overline{n+3}\right]\right\};$$

$$Y = \frac{1}{k(n-2)}\left(\frac{1}{q^{n-2}} - \frac{1}{p^{n-2}}\right)(\cos\gamma)^{n-1}\left\{\tan\gamma - \tfrac{1}{3}\cdot\frac{p-q}{p+q}(\alpha-\beta)\left[\overline{n-2}(\sec\gamma)^2 - \overline{n-1}\right] - \tfrac{1}{24}(\alpha-\beta)^2\tan\gamma\left[\overline{n-2}\,\overline{n+5}(\sec\gamma)^2 - \overline{n-1}\,\overline{n+3}\right]\right\};$$

$$T = \frac{1}{k(n-1)}\left(\frac{1}{q^{n-1}} - \frac{1}{p^{n-1}}\right)(\cos\gamma)^{n-1}\left\{1 - \frac{n-1}{6}\frac{p-q}{p+q}(\alpha-\beta)\tan\gamma - \frac{n-1}{24}(\alpha-\beta)^2\left[\overline{n+4}(\sec\gamma)^2 - \overline{n+3}\right]\right\};$$

and these values are still true to the fourth order, considering $\frac{p-q}{p+q}$ and $\alpha-\beta$ to be small quantities of the first order as before.

The angle which the chord of the arc makes with the axis of x becomes, in like manner—

$$\bar{\gamma} = \gamma + \tfrac{1}{3}\frac{p-q}{p+q}(\alpha-\beta) + \tfrac{1}{4}(\alpha-\theta)^2\tan\gamma,$$

which is true to the third order.

The above expressions for X and Y may be transformed by introducing this angle $\bar{\gamma}$ into them instead of γ, thus—

$$(\cos\bar{\gamma})^{n-1} = (\cos\gamma)^{n-1} - (n-1)(\cos\gamma)^{n-2}\sin\gamma\left[\tfrac{1}{3}\frac{p-q}{p+q}(\alpha-\beta) + \tfrac{1}{4}(\alpha-\beta)^2\tan\gamma\right]$$

$$= (\cos\gamma)^{n-1}\left\{1 - \frac{n-1}{3}\,\frac{p-q}{p+q}(\alpha-\beta)\tan\gamma - \frac{n-1}{4}(\alpha-\beta)^2(\tan\gamma)^2\right\}$$

Hence we find—

$$X = \frac{1}{k(n-2)}\left(\frac{1}{q^{n-2}} - \frac{1}{p^{n-2}}\right)(\cos\bar{\gamma})^{n-1}\left\{1 - \frac{n-1}{24}(\alpha-\beta)^2[\overline{n-2}\,(\sec\gamma)^2 - \overline{n-3}]\right\},$$

and

$$Y = X\tan\bar{\gamma} = \frac{1}{k(n-2)}\left(\frac{1}{q^{n-2}} - \frac{1}{p^{n-2}}\right)(\cos\bar{\gamma})^{n-2}\sin\bar{\gamma}\left\{1 - \frac{n-1}{24}(\alpha-\beta)^2[\overline{n-2}\,(\sec\gamma)^2 - \overline{n-3}]\right\};$$

or

$$X = \frac{1}{k(n-2}\left(\frac{1}{q^{n-2}} - \frac{1}{p^{n-2}}\right)(\cos\bar{\gamma})^{n-1}\,Q;$$

$$Y = \frac{1}{k(n-2)}\left(\frac{1}{q^{n-2}} - \frac{1}{p^{n-2}}\right)(\cos\bar{\gamma})^{n-2}\sin\bar{\gamma}\,Q;$$

Q being $= 1 - \dfrac{n-1}{24}(\alpha-\beta)^2[\overline{n-2}\,(\sec\gamma)^2 - \overline{n-3}]$.

Similarly, if

$$\bar{\gamma}' = \gamma + \tfrac{1}{6}\frac{p-q}{p+q}(\alpha-\beta) + \tfrac{1}{4}(\alpha-\beta)^2\tan\gamma,$$

we have

$$(\cos\bar{\gamma}')^{n-1} = (\cos\gamma)^{n-1} - (n-1)(\cos\gamma)^{n-2}\sin\gamma\left[\tfrac{1}{6}\frac{p-q}{p+q}(\alpha-\beta) + \tfrac{1}{4}(\alpha-\beta)^2\tan\gamma\right];$$

$$= (\cos\gamma)^{n-1}\left\{1 - \frac{n-1}{6}\,\frac{p-q}{p+q}(\alpha-\beta)\tan\gamma - \frac{n-1}{4}(\alpha-\beta)^2(\tan\gamma)^2\right\};$$

and therefore

$$T = \frac{1}{k(n-1)}\left(\frac{1}{q^{n-1}} - \frac{1}{p^{n-1}}\right)(\cos\bar{\gamma}')^{n-1}\left\{1 - \frac{n-1}{24}(\alpha-\beta)^2\left[\overline{n-2}\,(\sec\gamma)^2 - \overline{n-3}\right]\right\}$$

$$= \frac{1}{k(n-1)}\left(\frac{1}{q^{n-1}} - \frac{1}{p^{n-1}}\right)(\cos\bar{\gamma}')^{n-1}\,Q,$$

where Q has the same value as before.

Hence the values of X, Y, and T are as stated in my postscript to Mr. Niven's paper.

Although the method of finding the expressions for X and T given above, is perhaps the plainest and most straightforward that can be taken, the following leads to simpler operations.

Let $f(\phi) = u^l(\sec\phi)^{n+1}$.

Then $\displaystyle\int f(\phi)d\phi = \int u^l(\sec\phi)^{n+1}d\phi = \frac{g}{k}\int u^{l-n-1}\frac{du}{d\phi}d\phi$ by equation (1)

$$= \frac{g}{k(l-n)}u^{l-n} + \text{const.}$$

Hence

$$\int_\beta^\alpha f(\phi)d\phi = \frac{g}{k(l-n)}(p^{l-n} - q^{l-n})$$

Now let

$$F(\phi) = f(\phi)(\sec\phi)^m = u^l(\sec\phi)^{m+n+1},$$

then

$$F'(\phi) = f'(\phi)(\sec\phi)^m + mf(\phi)(\sec\phi)^m\tan\phi,$$

and

$$F''(\phi) = f''(\phi)(\sec\phi)^m + 2mf'(\phi)(\sec\phi)^m\tan\phi + mf(\phi)[m(\sec\phi)^m(\tan\phi)^2 + (\sec\phi)^{m+2}]$$

$$= f''(\phi)(\sec\phi)^m + 2mf'(\phi)(\sec\phi)^m\tan\phi + mf(\phi)[\overline{m+1}(\sec\phi)^{m+2} - m(\sec\phi)^m].$$

Hence, by the lemma,

$$\int_\beta^\alpha F(\phi)d\phi = (\alpha-\beta)\{F(\gamma) + \tfrac{1}{24}(\alpha-\beta)^2F''(\gamma)\}$$

$$= (\alpha-\beta)\left\{f(\gamma)(\sec\gamma)^m + \tfrac{1}{24}(\alpha-\beta)^2(\sec\gamma)^m[f''(\gamma) + 2mf'(\gamma)\tan\gamma + mf(\gamma)[\overline{m+1}(\sec\gamma)^2 - m]]\right\}$$

$$= (\alpha-\beta)(\sec\gamma)^m\left\{f(\gamma) + \tfrac{1}{24}(\alpha-\beta)^2[f''(\gamma) + 2mf'(\gamma)\tan\gamma + mf(\gamma)[\overline{m+1}(\sec\gamma)^2 - m]]\right\}.$$

But from above

$$\frac{g}{k(l-n)}(p^{l-n} - q^{l-n}) = \int^\alpha f(\phi)d\phi,$$

$$= (\alpha-\beta)\{f(\gamma) + \tfrac{1}{24}(\alpha-\beta)^2f''(\gamma)\}.$$

Hence, by division,

$$\int^\alpha F(\phi)d\phi \div \frac{g}{k(l-n)}(p^{l-n} - q^{l-n}) = (\sec\gamma)^m\left\{1 + \tfrac{1}{24}(\alpha-\beta)^2\left[\overline{2m}\frac{f'\gamma}{f\gamma}\tan\gamma + m(\overline{m+1}(\sec\gamma)^2 - m)\right]\right\},$$

It will be noticed that in this division the quantity $f''(\gamma)$ has disappeared.

Now, from above,

$$f(\phi) = u^l(\sec\phi)^{n+1},$$

and therefore

$$\frac{f'(\phi)}{f(\phi)} = l\frac{du}{ud\phi} + (n+1)\tan\phi,$$

and

$$\frac{f'(\gamma)}{f(\gamma)} = l\left(\frac{du}{ud\phi}\right)_0 + (n+1)\tan\gamma.$$

Hence

$$\int_\beta^\alpha \mathrm{F}(\phi)d\phi \div \frac{g}{k(l-n)}(p^{l-n} - q^{l-n}) = (\sec\gamma)^m\left\{1 + \tfrac{1}{24}(\alpha-\beta)^2\left[2lm\left(\frac{du}{ud\phi}\right)_0 \tan\gamma + 2m\,\overline{n+1}(\tan\gamma)^2 + m\left(\overline{m+1}(\sec\gamma)^2 - m\right)\right]\right\}$$

$$= (\sec\gamma)^m\left\{1 + \tfrac{1}{24}(\alpha-\beta)^2\left[2lm\left(\frac{du}{ud\phi}\right)_0 \tan\gamma + m(m+2n+3)(\sec\gamma)^2 - m(m+2n+2)\right]\right\}$$

Now make $m + n + 1 = 2$,

or $m = -(n-1)$, and we have

$$\int_\beta^\alpha u^l(\sec\phi)^2 \div \frac{g}{k(l-n)}(p^{l-n} - q^{l-n}) = (\cos\gamma)^{n-1}\left\{1 - \tfrac{1}{24}(\alpha-\beta)^2\left[2l(n-1)\left(\frac{du}{ud\phi}\right)_0 \tan\gamma + (n-1)(n+4)(\sec\gamma)^2 - (n-1)(n+3)\right]\right\}$$

In this make $l = 2$, and $l = 1$, successively, and we obtain the same expressions for X and T as before.

The case thus treated is not one of mere curiosity, but is practically important. From theoretical considerations, Newton concluded that the resistance of the air to the motion of projectiles is proportional to the square of the velocity, and very little progress has been made in the theory of the subject since his time. Experiments have shown that the relation between the velocity of a projectile and the resistance offered by the air to its motion is far from being so simple as that given by the theory. The most extensive and accurate series of such experiments which we have are those made by Mr. Bashforth by means of his chronograph, which measures with the greatest precision the times taken by the same projectile in passing over several successive arcs in the course of its flight. In a summary of his results for ogival-headed shot, struck with a radius of 1½ diameters, given in NATURE (vol. xxxiii. pp. 605, 606), Mr. Bashforth concludes that the resistance may be approximately represented by supposing it to vary as one power of the velocity when that velocity lies between certain limits, as another power when the velocity lies between certain other limits, and so on.

Thus, if v denote the velocity expressed in feet per second,

d the diameter of the shot in inches,

and w its weight in pounds,

and if $\frac{d^2}{w} = c$,

then, when v lies between 430 f.s. and 850 f.s.,

the resistance is nearly $= 61{\cdot}3\,c\left(\frac{v}{1000}\right)^2$;

when v lies between 850 f.s. and 1040 f.s.,

the resistance is nearly $= 74{\cdot}4\,c\left(\frac{v}{1000}\right)^3$;

when v lies between 1040 f.s. and 1100 f.s.,

the resistance is nearly $= 79{\cdot}2\,c\left(\frac{v}{1000}\right)^6$;

when v lies between 1100 f.s. and 1300 f.s.,

the resistance is nearly $= 108{\cdot}8\,c\left(\frac{v}{1000}\right)^3$;

and lastly, when v lies between 1300 f.s. and 2700 f.s.,

the resistance is nearly $= 141{\cdot}5\,c\left(\frac{v}{1000}\right)^2$.

Hence the resistance varies nearly as the square of the velocity both when the velocity is less than 850 f.s., and when it is greater than 1300 f.s., but the coefficient increases from 61·3 in the former case, to 141·5 in the latter. Also, the resistance varies nearly as the cube of the velocity, both when v lies between 850 f.s. and 1040 f.s., and also when it lies between 1100 f.s. and 1300 f.s., but the coefficient increases from 74·4 in the former to 108·8 in the latter case. Again, for velocities which are nearly equal to that of sound in air, the proportionate increase of the resistance is much greater than that of the velocity.

Mr. Bashforth remarks that the points of transition from one law of resistance to another, as stated above, are somewhat arbitrary, but that, if they were changed a little in either direction, the practical error would not be large.

Of course, if we had at our disposal much more numerous and still more accurate observations, it would be possible to represent the experimental results with any degree of exactness that might be desired, by subdividing the observations into a larger number of groups, so that the limiting velocities in any one group should be closer together, and that the change of the index of the power of the velocity in passing from one group to the next should be less abrupt.

J. C. ADAMS.

SOCIETIES AND ACADEMIES.

LONDON.

Chemical Society, December 19, 1889.—Dr. W. J. Russell, F.R.S., in the chair.—The following papers were read:—Frangulin, by Prof. T. E. Thorpe, F.R.S., and Mr. H. H. Robinson. The authors prepared the glucoside frangulin from the bark of the alder buckthorn (*Rhamnus frangula*), and find its formula to be $C_{22}H_{22}O_9$. On hydrolysis it yields a yellow product, $C_{15}H_{10}O_5$, which agrees in its properties with emodin, and a sugar which has the power of reducing Fehling's solution, and is not identical with dextrose.—Arabinon, the saccharon of arabinose, by Mr. C. O'Sullivan, F.R.S. The substance having an optical activity "well above $[\alpha]_j = 140$," obtained by the author by the hydrolysis of arabic acid, and described under the name of α-arabinose (Chem. Soc. Trans., 1884, 55), yields arabinose on hydrolysis, and appears to bear to this carbohydrate a relation similar to that which saccharon (cane sugar) bears to dextrose: the author therefore terms it arabinon. It has the formula $C_{10}H_{18}O_9$, and on hydrolysis gives a yield of arabinose agreeing very closely with that required by the equation $C_{10}H_{18}O_9 + H_2O = 2C_5H_{10}O_5$. As yet it has not been obtained in a crystalline state; it has a specific rotatory power of $[\alpha]_D = 198°{\cdot}8$, and 100 parts have the same cupric reducing power as 58·8 parts of dextrose.—On the identity of cerebrose and galactose, by Mr. H. T. Brown, F.R.S., and Dr. G. H. Morris. The authors give the results of an examination of a specimen of cerebrose, prepared from phrenosin, which was placed in their hands early in 1888 by Dr. Thudichum, who first isolated and crystallized this substance. They show that its specific rotatory power, cupric reducing power, and molecular weight as determined by Raoult's method, are identical with those of galactose, thus confirming the recent work of Thierfelder, *Zeit. Physiol. Chem.*, 14, 209) who has proved the sugar produced by the action of acid on cerebrin to be identical with galactose. In the discussion which followed the reading of the paper, Dr. Thudichum said that phrenosin, $C_{41}H_{79}NO_8$, consisted of the sugar now shown to be identical with galactose, $C_6H_{12}O_6$, of neurostearic acid, $C_{18}H_{36}O_2$, an isomeride of stearic acid, fusing at 84°, and of sphingosine, an